Skewness Premium with Lévy Processes

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- Motivation
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- Lévy processes
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- Duality and Symmetry
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• Conclusions
Motivation

- Observed moneyness biases in American call and put options
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- S&P500 options traded on CMEX
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- American Foreign currency call options traded in Philadelphia Stock Exchange
**Motivation**

- Observed moneyness biases in American call and put options
- S&P500 options traded on CMEX
- American Foreign currency call options traded in Philadelphia Stock Exchange
- The Biases are not in the same direction, nor are they constant over time.
Some facts

- Out-of-the-money (OTM) Calls pays only if the asset price rises above the Call’s exercise price while OTM Puts pay off only if asset price falls below the Put’s exercise price.
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- Call and Put prices directly reflects characteristics of the upper and lower tails of the risk neutral distribution.
Some facts

- Out-of-the-money (OTM) Calls pays only if the asset price rises above the Call’s exercise price while OTM Puts pay off only if asset price falls below the Put’s exercise price.
- Call and Put prices directly reflects characteristics of the upper and lower tails of the risk neutral distribution.
- Then relative prices of OTM options will reflect the skewness of the risk neutral distribution.
Put-Call relationship

Put-Call Parity:

\[ p + S = c + X e^{-rT} \]
Put-Call relationship

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Just for European Options!
Put-Call relationship

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Put-Call Duality:

\[ C(\cdot) = P(\cdot) \]
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European and American Options!
Put-Call relationship

Put-Call Parity:

\[ p + S = c + X e^{-rT} \]

Just for European Options!  Same Strike

Put-Call Duality:

\[ C(\cdot) = P(\cdot) \]

European and American Options!  Different Strike
From Duality

Call Options $x\%$ out-of-the-money are priced exactly $x\%$ higher than the corresponding OTM put:
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$$C(F, T; K_c) = (1 + x)P(F, T; K_p), \quad x > 0$$
Call Options \( x\% \) out-of-the-money are priced exactly \( x\% \) higher than the corresponding OTM put:

\[
C(F, T; K_c) = (1 + x) P(F, T; K_p), \quad x > 0
\]

Where \( K_c = F(1 + x) \) and \( K_p = F/(1 + x) \).
Call Options $x\%$ out-of-the-money are priced exactly $x\%$ higher than the corresponding OTM put:

$$C(F, T; K_c) = (1 + x)P(F, T; K_p), \quad x > 0$$

Where $K_c = F(1 + x)$ and $K_p = F/(1 + x)$.

Bates’ $x\%$ rule!
Skewness Premium (SK)

David S. Bates

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David S. Bates

- The Skewness Premium: Option Pricing Under Asymmetric Processes, Advances in Futures and Options Research 9, 1997, 51-82
The Skewness Premium (SK)
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- The Skewness Premium: Option Pricing Under Asymmetric Processes, Advances in Futures and Options Research 9, 1997, 51-82
- For which parameters $SK = \frac{C}{P} - 1 \leq 0$?
Option Prices on S&P500 in 08/31/2006.
Some facts: OTM options S&P500-Aug 31/06. T=Sept 15/06, F=1303.82

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Some facts: OTM options S&P500-Aug 31/06. T=Sept 15/06, F=1303.82

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Skewness Premium (SK)

- OTM options: Usually, $x_{obs} < x$. That means $\frac{c}{p} - 1 < x$. 

Asset returns negatively skewed.
Skewness Premium (SK)

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Skewness Premium (SK)

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- ITM options: Usually, \( x_{obs} > x \). That means \( \frac{c}{p} - 1 > x \).
- Asset returns negatively skewed.
Theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process.
Contribution

- Theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process.
- Simply diagnostic for judging which distributions are consistent with observed option prices.
Lévy Processes

Consider a stochastic process $X = \{X_t\}_{t \geq 0}$, defined on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$. We say that $X = \{X_t\}_{t \geq 0}$ is a Lévy Process if:

- the process has paths of right-continuous left-limits (RCLL) and $X_0 = 0$,
- the increments are independent,
- the distribution of the increment $X_{t_1} - X_{t_2}$ is homogenous in time, that is, depends just on the difference $t_1 - t_2$.
Lévy Processes

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- $X$ has paths RCLL
- $X_0 = 0$, and has independent increments, given $0 < t_1 < t_2 < \ldots < t_n$, the r.v.

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independents.
Lévy Processes

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  \[ X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \]
  are independents.
- The distribution of the increment $X_t - X_s$ is homogenous in time, that is, depends just on the difference $t - s$. 
Lévy-Khintchine Formula

A key result in the theory of Lévy Processes is the Lévy-Khintchine formula, that computes de characteristic function of $X_t$ como:

$$E(e^{zX_t}) = e^{t\psi(z)}$$
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$$E(e^{zX_t}) = e^{t\psi(z)}$$

Where $\psi$ is called characteristic exponent, and is given by:

$$\psi(z) = az + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy1_{|y|<1})\Pi(dy),$$

where $b$ and $\sigma \geq 0$ are real constants, and $\Pi$ is a positive measure in $\mathbb{R} - \{0\}$ such that

$$\int (1 \wedge y^2)\Pi(dy) < \infty,$$

called the Lévy measure. The triplet $(a, \sigma^2, \Pi)$ is the characteristic triplet.
Model

Consider a market with two assets given by

\[ S^1_t = e^{X_t}, \quad \text{and} \quad S^2_t = S^2_0 e^{rt} \]

where \((X)\) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take \(S^1_0 = 1\).
**Model**

Consider a market with two assets given by

\[ S^1_t = e^{X_t}, \quad \text{and} \quad S^2_t = S^2_0 e^{rt} \]

where \((X)\) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take \(S^1_0 = 1\).

In this model we assume that the stock pays dividends with constant rate \(\delta \geq 0\), and that the given probability measure \(\mathbb{Q}\) is the chosen equivalent martingale measure.
Duality

Denote by $\mathcal{M}_T$ the class of stopping times up to a fixed constant time $T$, i.e:

$$\mathcal{M}_T = \{\tau : 0 \leq \tau \leq T, \tau \text{ stopping time w.r.t } F\}$$

for the finite horizon case and for the perpetual case we take $T = \infty$ and denote by $\overline{\mathcal{M}}$ the resulting stopping times set. Then, for each stopping time $\tau \in \mathcal{M}_T$ we introduce

$$c(S_0, K, r, \delta, \tau, \psi) = \mathbb{E} e^{-r\tau} (S_\tau - K)^+, \quad (1)$$

$$p(S_0, K, r, \delta, \tau, \psi) = \mathbb{E} e^{-r\tau} (K - S_\tau)^+. \quad (2)$$
Duality

For the American finite case, prices and optimal stopping rules \( \tau_c^* \) and \( \tau_p^* \) are defined, respectively, by:

\[
C(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} \mathbb{E} e^{-r\tau} (S_\tau - K)^+
\]

\[
= \mathbb{E} e^{-r\tau_c^*} (S_{\tau_c^*} - K)^+ \quad (3)
\]

\[
P(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} \mathbb{E} e^{-r\tau} (K - S_\tau)^+
\]

\[
= \mathbb{E} e^{-r\tau_p^*} (K - S_{\tau_p^*})^+, \quad (4)
\]
Duality

And for the American perpetual case, prices and optimal stopping rules are determined by

\[
\overline{C}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \mathcal{M}} \mathbb{E} e^{-r\tau} (S_\tau - K)^+ 1_{\{\tau < \infty\}}
\]

\[
= \mathbb{E} e^{-r\tau^*_c} (S_{\tau^*_c} - K)^+ 1_{\{\tau < \infty\}},
\]  

(5)

\[
\overline{P}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \mathcal{M}} \mathbb{E} e^{-r\tau} (K - S_\tau)^+ 1_{\{\tau < \infty\}}
\]

\[
= \mathbb{E} e^{-r\tau^*_p} (K - S_{\tau^*_p})^+ 1_{\{\tau < \infty\}}.
\]  

(6)
**Put-Call Duality**

**Lemma 0.1 (Duality).** Consider a Lévy market with driving process $X$ with characteristic exponent $\psi(z)$. Then, for the expectations introduced in (1) and (2) we have

$$c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}),$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2z^2 + \int(e^{zy} - 1 - zh(y))\tilde{\Pi}(dy)$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\begin{cases} 
\tilde{a} &= \delta - r - \sigma^2/2 - \int(e^y - 1 - h(y))\tilde{\Pi}(dy), \\
\tilde{\sigma} &= \sigma, \\
\tilde{\Pi}(dy) &= e^{-y}\Pi(-dy).
\end{cases}$$
**Duality**

**Corollary 0.1 (European Options).** For the expectations introduced in (1) and (2) we have

\[ c(S_0, K, r, \delta, T, \psi) = p(K, S_0, \delta, r, T, \tilde{\psi}), \tag{10} \]

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma.

**Corollary 0.2 (American Options).** For the value functions in (3) and (4) we have

\[ C(S_0, K, r, \delta, T, \psi) = P(K, S_0, \delta, r, T, \tilde{\psi}), \tag{11} \]

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma.
Corollary 0.3 (Perpetual Options). For prices of Perpetual Call and Put options in (5) and (6) the optimal stopping rules have, respectively, the form

\[ \tau_c^* = \inf \{ t \geq 0 : S_t \geq S_c^* \}, \]
\[ \tau_p^* = \inf \{ t \geq 0 : S_t \leq S_p^* \}. \]

where the constants \( S_c^* \) and \( S_p^* \) are the critical prices. Then, we have

\[ \overline{C}(S_0, K, r, \delta, \psi) = \overline{P}(K, S_0, \delta, r, \tilde{\psi}), \]

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma. Furthermore, when \( \delta > 0 \), for the optimal stopping levels, we obtain the relation

\[ S_c^* S_p^* = S_0 K. \]
Dual markets

Given a Lévy market with driving process characterized by $\psi$, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t \geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \quad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t \geq 0}$, modelled by

$$\tilde{S}_t = Ke^{\tilde{X}_t}, \quad \tilde{S}_0 = K > 0,$$

where $\tilde{X}_t = -X_t$ is a Lévy process with characteristic exponent under $\tilde{Q}$ given by $\tilde{\psi}$ in (8). The process $\tilde{S}_t$ represents the price of $KS_0$ dollars measured in units of stock $S$. 
Symmetric markets

Let's define symmetric markets by

\[
\mathcal{L}(e^{-(r-\delta)t+X_t} \mid \mathbb{Q}) = \mathcal{L}(e^{-(\delta-r)t-X_t} \mid \tilde{\mathbb{Q}}),
\]  

meaning equality in law.

A necessary and sufficient condition for (14) to hold is

\[
\Pi(dy) = e^{-y}\Pi(-dy),
\]  

This ensures \(\tilde{\Pi} = \Pi\), and from this follows

\[
a - (r - \delta) = \tilde{a} - (\delta - r)
\]

, giving (14), as always \(\tilde{\sigma} = \sigma\).
\textit{Bates’ $x\%$-Rule}

If the call and put options have strike prices $x\%$ out-of-the-money relative to the forward price, then the call should be priced $x\%$ higher than the put.
**Bates’ $x\%$-Rule**

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If $r = \delta$, we can take the future price $F$ as the underlying asset in Lemma 1.
**Bates’ \( x\% \)-Rule**

If the call and put options have strike prices \( x\% \) out-of-the-money relative to the forward price, then the call should be priced \( x\% \) higher than the put.

If \( r = \delta \), we can take the future price \( F \) as the underlying asset in Lemma 1.

**Corollary 0.6.** Take \( r = \delta \) and assume (15) holds, we have

\[
\mathcal{C}(F_0, K_c, r, \tau, \psi) = x \mathcal{P}(F_0, K_p, r, \tau, \psi),
\]

where \( K_c = xF_0 \) and \( K_p = F_0/x \), with \( x > 0 \).
**Diffusions with jumps**

Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

\[
\Pi(dy) = \lambda \frac{1}{\delta \sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy,
\]

and is direct to verify that condition (15) holds if and only if \(2\mu + \delta^2 = 0\). This result was obtained by Bates (1997) for future options.
Diffusions with jumps

Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

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and is direct to verify that condition (15) holds if and only if $2\mu + \delta^2 = 0$. This result was obtained by Bates (1997) for future options.

That result is obtained as a particular case, if we replace the future price as being the underlying asset, when $r = \delta$ in Lemma 1.
**Lévy Processes**

We restrict to Lévy markets with jump measure of the form

\[ \Pi(dy) = e^{\beta y} \Pi_0(dy), \]

where \( \Pi_0(dy) \) is a symmetric measure, i.e. \( \Pi_0(dy) = \Pi_0(-dy) \), everything with respect to the risk neutral measure \( \mathbb{Q} \).
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As a consequence of (15), market is symmetric if and only if \( \beta = -1/2 \).
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As a consequence of (15), market is symmetric if and only if \( \beta = -1/2 \).

In view of this, we propose to measure the asymmetry in the market through the parameter \( \beta + 1/2 \). When \( \beta + 1/2 = 0 \) we have a symmetric market.
Esscher Transform

We can obtain an Equivalent Martingale Measure by

\[ dQ_t = \frac{e^{\theta X_t}}{E_P e^{\theta X_t}} dP_t \]
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There is a \( \theta \) such that the discounted price process is a martingale respect to \( Q \).

As a consequence:

\[ \beta_Q = \beta_P + \theta \]
Example 1

Consider the Generalized Hyperbolic Distributions, with Lévy measure:

$$\Pi(dy) = e^{\beta y} \frac{1}{|y|} \left( \int_0^\infty \frac{\exp \left( -\sqrt{2z + \alpha^2 |y|} \right)}{\pi^2 z \left( J_\lambda^2 (\delta \sqrt{2z}) + Y_\lambda^2 (\delta \sqrt{2z}) \right)} dz + 1_{\{\lambda \geq 0\}} \lambda e^{-\alpha |y|} \right) dy$$

where $\alpha, \beta, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta| < \alpha$, and $\delta > 0$; and $J_\lambda, Y_\lambda$ are the Bessel functions of the first and second kind.
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Eberlein and Prause (1998): German Stocks
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\[ \beta_P = -0.0035 \quad \text{and} \quad \beta_Q = 80.65. \]
**Parametros Estimados GH**

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<th>Sample</th>
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</table>
Example 2

Consider the Meixner distribution, with Lévy measure:

\[ \Pi(dy) = c \frac{e^{\frac{b}{a}y}}{y \sinh(\frac{\pi y}{a})} dy, \]

where \( a, b \) and \( c \) are parameters of the Meixner density, such that \( a > 0, -\pi < b < \pi \) and \( c > 0 \). Then \( \beta_P = b/a \).
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<table>
<thead>
<tr>
<th>Index</th>
<th>( \hat{a} )</th>
<th>( \hat{b} )</th>
<th>( \theta )</th>
<th>( \beta_Q + 1/2 )</th>
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Example 3

This CGMY model, proposed by Carr et al. (2002) is characterized by $\sigma = 0$ and Lévy measure given by (28), where the function $p(y)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}} e^{-\alpha|y|}.$$

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where $C, G, M, Y$ are the parameters of the model.
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Values of $\beta = (G - M)/2$ are obtained for different assets under the market risk neutral measure and in the general situation, the parameter $\beta$ is negative and less than $-1/2$. 
**Implied volatility**

- Any model satisfying (15) must have identical Black-Scholes implicit volatilities for calls and puts with strikes \( \ln(K_c/F) = \ln x = -\ln(K_p/F) \), with \( x > 0 \) arbitrary.
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- That is, the volatility smile curve is symmetric in the moneyness $\ln(K/F)$. 
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- That is, the volatility smile curve is symmetric in the moneyness $\ln(K/F)$.

- By put-call parity, European calls and puts with same strike and maturity must have identical implicit volatilities.
Skewness Premium (SK)

The \( x\% \) Skewness Premium is defined as the percentage deviation of \( x\% \) OTM call prices from \( x\% \) OTM put prices.
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$$SK(x) = \frac{c(S, T; X_c)}{p(S, T; X_p)} - 1, \quad \text{for European Options,} \quad (20)$$

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Skewness Premium (SK)

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- Constant Elasticity of Variance (CEV), include arithmetic and geometric Brownian motion.
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- Jump-diffusion processes, the benchmark model is the Merton’s (1976) model.
Some results

For European options in general and for American options on futures, the SK has the following properties for the above distributions.

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Now in equation (21) consider

\[ X_p = F(1 - x) < F < F(1 + x) = X_c, \quad x > 0. \]
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For in-the-money options \( (x < 0) \), the propositions are reversed.

 Calls \( x\% \) in-the-money should cost \( 0\% - x\% \) less than puts \( x\% \) in-the-money.
Some results

**Theorem 0.1.** Take \( r = \delta \) and assume that in the particular case (28), if \( \beta \geq -1/2 \), then

\[
c(F_0, K_c, r, \tau, \psi) \geq (1 + x) p(F_0, K_p, r, \tau, \psi),
\]

where \( K_c = (1 + x) F_0 \) and \( K_p = F_0/(1 + x) \), with \( x > 0 \).
Conclusions

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- Time-Changed Lévy Processes
- Other Derivatives
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